Ontological Shape, Propositional Category

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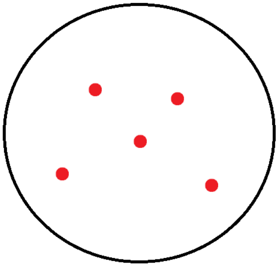
Overview

- Want to rigorously define concepts
- But also organize them and make statements about them
- Category theory itself allows us to organize concepts
- Specific Categories give us a means of making statements
- Ontologies are meant to organize and make statements about concepts
- But as "abstract objects" we lose meaning of the full definition of a concept
- So we we will define an "Ontological Expansion" to help define concepts

All in the language of Category Theory

Organizing Concepts: Set Theory

Modern math has given us pretty ubiquitous ways of organizing ideas:
For example set theory:



To speak of "elements"

Organizing Concepts: Graph Theory

Modern math has given us pretty ubiquitous ways of organizing ideas:
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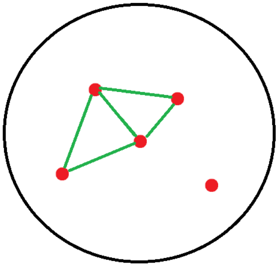
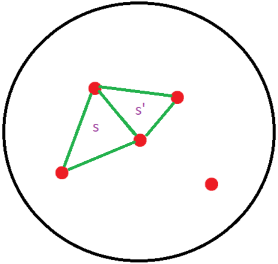


To speak of "elements & relations"

Organizing Concepts: Simplicial Set Theory

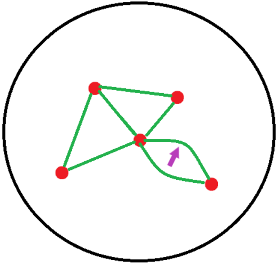
Modern math has given us pretty ubiquitous ways of organizing ideas:
For example Simplicial Set theory:



To speak of "elements, relations & composition"

Organizing Concepts: Globular Set Theory

Modern math has given us pretty ubiquitous ways of organizing ideas:
For example Globular Set theory:



To speak of "elements, relations & higher relations"

Organizing Concepts: Ontological Shape

All of these concepts have one thing in common: They are small functors into Set

- Set: $* \to Set$
- Graph: $(\Delta|^2)^{op} \to set$
- sSet: $\Delta^{op} \to Set$
- gSet: $G^{op} \to Set$
- (cubical sets, cosimplicial sets, etc.)

Regardless they all show up as contravariant functors $\Delta^{op} \to Set$

Definition: Ontological Shape

An "Ontological Shape" is just a small category Δ

Examples: $\Delta \downarrow X$

For a space X, consider $\Delta \downarrow X[n] = \{\Delta^n \to X\}$

- The "concepts" in this sSet are points
- The "relations" are paths
- The compositions are homotopy concatinations

  ...

Examples: *CGHaus*

Consider the globular set of compactly generated hausdorff spaces CGHaus

- The "concepts" in this gSet are spaces
- the "relations" are continuous maps
- "higher relations" are homotopies

  ...

Propositional Category: Set

Each of these models enjoy statements from set theory

- We can say "x is an element of S"
  - "X is a compactly generated hausdorff space"
  - "(1,2) is a point in $\mathbb{R}^2$"
  - "γ is a path in $\mathbb{R}^2$"
- We can say "A is a subset of S", hence logical deduction
  - "If X is compact, then it is compactly generated"
- We can also use existential, or universal quantifiers
  - "For all x,y in $\mathbb{R}^2$, there is a path $\gamma : x \rightsquigarrow y$"

Propositional Category: Other Examples

Of course other categories give us other statements:

- $\mathfrak{Mod}_K$ "N is a submodule of M"
- Reimann "$\gamma : x \rightsquigarrow y$ is a geodesic"
- $\mathfrak{Ch}_K$ "x is the boundary of y"

Or we may ask for a 2-category

- $\mathfrak{Cat}$ "$\prod_x F(x)$ is universal"
- $\mathfrak{Ab}$ "$A \to B \to C$ is exact"

Definition: Propositional Category

A "Propositional Category" $\mathfrak{C}$ is just a category

Basic Ontology

The Ontological shape Δ organizes our concepts
The Propositional Category $\mathfrak{C}$ allows us to make statements

Definition : Basic $(\Delta, \mathfrak{C})$-Ontology

A Basic $(\Delta, \mathfrak{C})$-Ontology is a functor $\Sigma : \Delta^{op} \to \mathfrak{C}$

- When Δ and $\mathfrak{C}$ are clear, we will just say "basic ontology"
- When $\Sigma : \Delta^{op} \to \mathfrak{C}$ has a name (such as "simplicial set") we will stick to that name

Rigorous Definition of Objects

Using Basic Ontologies we can:

- Organize concepts via Δ
- Make Statements about them in $\mathfrak{C}$

But a good notion of ontology should also include how to define it's objects **unambiguously**.

We will detail this issue with an example.

Rigorous Definition of Objects: Example $\mathfrak{Top}$

Consider the simplicial set of topological spaces:
That is, the nerve $N(\mathfrak{Top})$ (bounded above by an cardinal)

- Objects are spaces X
- n-Simplecies are compositions of maps

If I say "Consider $\mathbb{R}^2$", you may know what I mean...
but what if I am lying to you?
That is, what if I really mean $\mathbb{D}^2$, or ΩS^{42}?

Rigorous Definition of Objects: Example $\mathfrak{Top}$

In $\mathfrak{Top}$, I have a way of dealing with this ambiguity:
Namely, check the **isomorphism class** of what I labeled "$\mathbb{R}^2$"

Rigorous Definition of Objects: Example $\mathfrak{Top}$

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Namely, check the **isomorphism class** of what I labeled "$\mathbb{R}^2$"
but this isn't always available, what if I give you the simplicial set:

$$\{\mathbb{R}^2 \xrightarrow{4x} \mathbb{R}^2\}$$

In this simplicial set, I can no longer consider isomorphism classes.
However, it seems I have embedded this simplicial set into $\mathfrak{Top}$
so in $\mathfrak{Top}$ there I can check $\mathbb{R}^2$'s isomorphism class

Rigorous Defintion of Objects

This is great, assuming we have this embedding

$$\{\mathbb{R}^2 \xrightarrow{4x} \mathbb{R}^2\} \hookrightarrow \mathfrak{Top}$$

but in other cases we might not:

$$\{\text{France} \xrightarrow{\text{is in}} \text{EU}\}$$

Instead of isomorphism class, we define concepts by their **internals**

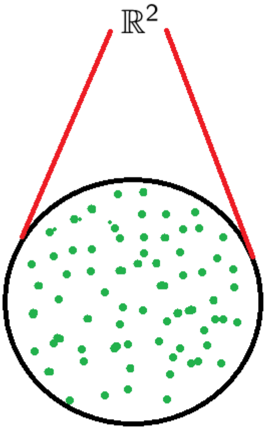
Ontological Expansion: Example $\mathbb{R}^2$

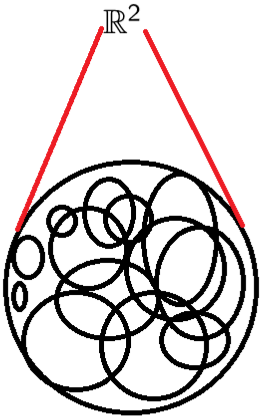
For example, the topological space $\mathbb{R}^2$ is not defined by it's label but rather:

- Its Points
- Its Open Sets

These will reveal themselves as two Ontological Expansions

Ontological Expansion: Example $\mathbb{R}^2$





Submorphism

In the example above, we expanded R^2 in two different ways:

- into a "collection" of points
- and a "collection" of open sets

We've expanded the objects (of $\mathfrak{Top}$), but now we want to expand the simplecies

We will do this by definining an **n-submorphism**

Submorphism

Intuitively, an n-submorphism is a collection of k-simplecies, $k \geq n$, closed under relevant face maps.

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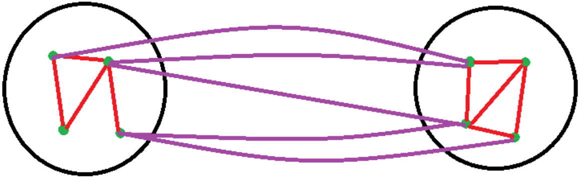
- 0-submorphism is just a sub-simplicial set
- Here is a picture of a 1-submorphism:



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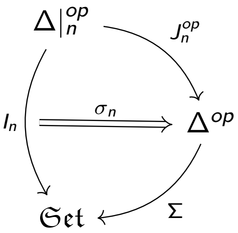
and then consider natural transformations into it

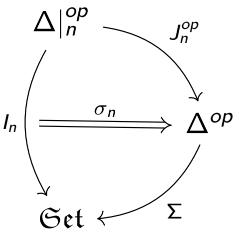


Submorphism Definition

Definition: n-submorphism

An <u>n-submorphism</u> of a simplicial set Σ is a natural transformation

$$\sigma_n : I_n \to J_n^*(\Sigma)$$



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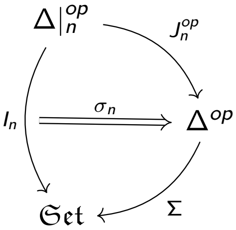

Again, intuitively, one can think of a submorphism as a collection of subset $\sigma_n = \{S_k \subseteq \Sigma[k]\}_{k \geq n}$

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Informally, σ_{n-1} is an i-face of σ_n if:

$$\forall s \in n^*(\sigma_n), \quad f_i(s) \in (n-1)^*(\sigma_{n-1})$$

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Definition: sm(Σ)

Let Σ be a simplicial set, then $sm(\Sigma)$ are the submorphisms in Σ

(In reality it is a functor $sm(\Sigma) : \Delta^{op} \to \mathfrak{Set}_P$)

Ontological Expansion: Definition

We are now ready for a definition of an ontological expansion:

Definition: Ontological Expansion (of a simplicial sets)

Let $\Sigma, \Sigma' : \Delta^{op} \to \mathfrak{Set}$ simplicial sets, An ontological expansion is a natural transformation

$$O : \Sigma \to sm(\Sigma')$$

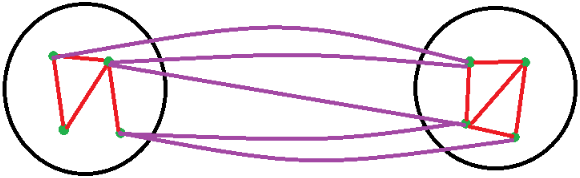
Ontological Expansion: Intuition

for an ontological expansion $O : \Sigma \to sm(\Sigma')$:

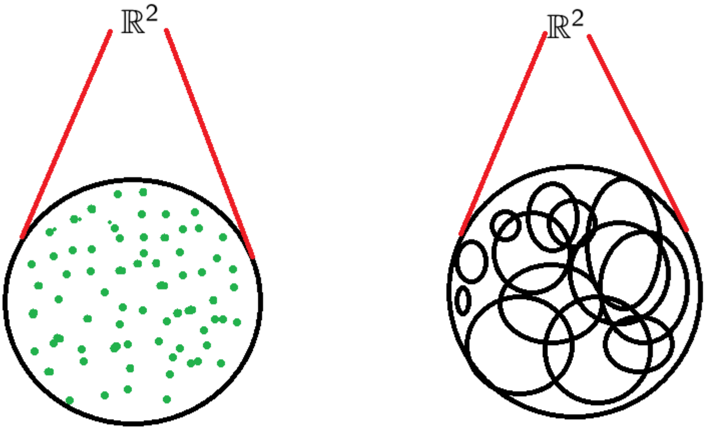
- For every 0-simplex, we expand to a sub-simplicial set
- For every 1-simplex, we expand to a 1-submorphism
- The faces of this 1-simplex expand to faces of the 1-submorphism

...

$$A \to B$$



Ontological Expansion: Example $\mathbb{R}^2$



Ontological Expansion: Example $U : \mathfrak{Top} \to sm(\mathfrak{Set})$

Consider the simplicial sets $\mathfrak{Top}$ and $\mathfrak{Set}$
We will define the ontological expansion $U : \mathfrak{Top} \to sm(\mathfrak{Set})$:

- $U(X) = \{*_x\}$
- $U(f) = \{*_x \to *_{f(x)}\}$
- $U(g \circ f = h) = \{*_x \to *_{f(x)} \to *_{g(f(x))} = *_x \to *_{h(x)}\}$

  ...

This is an ontological expansion:

- U(X),U(Y) are faces of U(f)
- U(f),U(g), U(h) are faces of $U(g \circ f = h)$

Ontological Expansion:Example $U : \mathfrak{Top} \to sm(\mathfrak{Top})$

Consider the simplicial set $\mathfrak{Top}$
We will define the ontological expansion $U : \mathfrak{Top} \to sm(\mathfrak{Top})$:

- $O(X) = \{U \subseteq X \text{ open}\}$
- $O(f) = \{f|_U : U \to V \ : \ V \supseteq f(U)\}$
- $O(g \circ f = h) =$

$$\{\ g|_V \circ f|_U = h|_U : U \to W \ : \ V \supseteq f(U),\ W \supseteq g(f(U))\ \}$$

O(X),O(Y) are faces of O(f), etc...

Summary

We are looking for a notion of ontology that:

- Allows us to organize concepts
- Allows us to make statements about concepts
- Allows us to define concepts formally

This is achieved by:

- Ontological Shape Δ
- Propositional Category $\mathfrak{C}$
- Ontological Expansions $O : \Sigma \rightarrow sm(\Sigma')$

Definition: Ontology

So our proposed defintion of ontology is:

Definition*: Ontology

An Ontology is a basic $(\Delta, \mathfrak{C})$-Ontology Σ, with a collection of Ontological Expansions

$$\{O_i : \Sigma \to sm(\mathcal{T}_i)\}$$

(If this reminds you of a site, this is a good thing. A (small) site is an ontology whose expansions are collections of coverings)